Exact String-Like Solutions in Conformal Gravity

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Abstract

The cylindrically-symmetric static vacuum equations of Conformal Gravity are solved for the case of additional boost symmetry along the axis. We present the complete family of solutions which describe the exterior gravitational field of line sources in Conformal Gravity. We also analyze the null geodesics in these spaces.

1 Introduction

Conformal Gravity (CG) [1] was proposed as a possible alternative to Einstein gravity ("GR"), which may supply the proper framework for a solution to some of the most annoying problems of theoretical physics like those of the cosmological constant, the dark matter and the dark energy.

The gravitational field in CG is still minimally coupled to matter, but the dynamical basis is different: it is obtained by replacing the Einstein-Hilbert action with the Weyl action based on the Weyl (or *conformal*) tensor $C_{\kappa\lambda\mu\nu}$ defined as the totally traceless part of the Riemann tensor:

$$C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - \frac{1}{2}(g_{\kappa\mu}R_{\lambda\nu} - g_{\kappa\nu}R_{\lambda\mu} + g_{\lambda\nu}R_{\kappa\mu} - g_{\lambda\mu}R_{\kappa\nu}) + \frac{R}{6}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}), \tag{1.1}$$

so the gravitational Lagrangian is

$$\mathcal{L}_g = -\frac{1}{2\alpha}C_{\kappa\lambda\mu\nu}C^{\kappa\lambda\mu\nu} \tag{1.2}$$

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where α is a dimensionless parameter. The gravitational field equations take the following form:

$$W_{\mu\nu} = \frac{\alpha}{2} T_{\mu\nu} \tag{1.3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and $W_{\mu\nu}$ is the Bach tensor given by

$$W^{\mu\nu} = R_{\kappa\lambda} C^{\kappa\mu\lambda\nu} - 2\nabla_\kappa \nabla_\lambda C^{\kappa\mu\lambda\nu}, \tag{1.4}$$

or in terms of the Riemann and Ricci components by:

$$W_{\mu\nu} = \frac{1}{3}\nabla_\mu \nabla_\nu R - \nabla_\lambda \nabla^\lambda R_{\mu\nu} + \frac{1}{6}(R^2 + \nabla_\lambda \nabla^\lambda R - 3R_{\kappa\lambda}R^{\kappa\lambda})g_{\mu\nu} + 2R^{\kappa\lambda}R_{\mu\kappa\nu\lambda} - \frac{2}{3}RR_{\mu\nu}. \tag{1.5}$$

It was suggested (see [1] and references therein) that while CG agrees with Newtonian gravity in Solar System scales, it further produces a linearly growing potential that could explain galactic rotation curves without invoking dark matter. It was further argued that accelerating cosmological solutions of CG describe naturally the accelerated expansion of the universe thus removing the need for dark energy.

On the other hand, CG has been criticized from several aspects both phenomenological and formal. Arguments in favor of the need of dark matter come from observations of the unusual object called "bullet cluster" [2, 3] whose dynamics seems very difficult to understand without assuming a weekly interacting dark component.

More specifically, several authors claim that predictions in the weak field limit of CG disagree with solar system observations [4], yield wrong light deflection [5] and that the exterior solutions cannot be matched to any source with a "reasonable" mass distribution [6]. Other authors find evidence for tachyons or ghosts [7] or raise the fact that only null geodesics are physically meaningful in this theory since the "standard" point particle Lagrangian is not conformally-invariant [8, 9].

Counter arguments to some of these objections were also published [10, 11], as well as possible ways [9] out of some of the difficulties and the matter is, to our view, still waiting for a consensus.

It is therefore very much required to investigate further the predictions and consequences of CG in its purely tensorial formulation as well as in its scalar tensor extension as much as possible.

In this work we concentrate on cylindrically-symmetric static vacuum solutions with the aim of clarifying further the properties of string-like solutions in CG [12]. Cosmic strings [13] are a typical outcome in any field theory which describes matter in the very early universe, thus serving very well the purpose of testing the implications of CG. The main result reported here is the full family of the static cylindrically-symmetric vacuum solutions of CG which represent the external gravitational field of localized line sources.

2 Cylindrically-Symmetric Equations

The general static cylindrically symmetric line-element has the form:

$$ds^2 = B^2(r)dt^2 - M^2(r)dr^2 - L^2(r)d\varphi^2 - K^2(r)dz^2 \tag{2.1}$$

In order to find solutions for this system, we have to fix the arbitrariness of the radial coordinate and the arbitrary rescaling of the metric due to the conformal symmetry. We will also limit our solutions to those exhibiting boost symmetry along the string direction (z), as for ordinary cosmic strings, i.e. $K(r) = B(r)$. This leaves one independent metric component. So only one of the gravitational field equations (1.3) (with $T^\mu_\nu = 0$) has to be solved. We may therefore choose to solve the lower order rr equation, i.e. $W^r_r = 0$.

Several special solutions are already known in explicit form. First of all, all the thin string (line source) solutions of GR with either a vanishing or non-vanishing cosmological constant [14] which solve for $r > 0$

$$R^\mu_\nu = \frac{\Lambda}{4}\delta^\mu_\nu \tag{2.2}$$

satisfy also the CG vacuum equations $W_{\mu\nu} = 0$. This is obvious by direct substitution of (2.2) in Eq. (1.5).

One special member of the $\Lambda < 0$ family is the AdS soliton [15] (see also [16, 17, 18]) which is a cylindrically-symmetric regular solution of the same equation (2.2) and is therefore distinct from AdS (anti-de Sitter) space. More recently, we have discovered two families of very simple exact solutions [12] during a mainly numerical study of cylindrical solutions of the Abelian Higgs model coupled to CG. However, we were unable to integrate the equations analytically at the time.

Here we present a reduction of the vacuum equations for static cylindrically-symmetric solutions to a single first order non-linear equation which may be solved by a straightforward quadrature.

We have therefore first to complete the gauge fixing. Since we would like it to be consistent with the symmetric vacuum solutions [i.e. (A)dS spaces] and with the AdS soliton, it should respect the asymptotic condition

$$R^\mu_\nu \to \frac{\kappa}{4}\delta^\mu_\nu \quad \text{for} \ \ r \to \infty \tag{2.3}$$

where κ is a real parameter. Note however, that a constant Ricci scalar is not a "gauge invariant" concept in CG; it is only a matter of convenience which can be obtained by a proper gauge choice. The corresponding invariant condition is that the Weyl tensor will vanish asymptotically, that is spacetime becomes conformally flat asymptotically. These restrictions will simplify considerably

the very cumbersome expressions of the components of the Bach tensor and will enable a clear physical picture.

The simplest gauge choice is $B(r) = K(r) = M(r) = 1$ with $L(r)$ as a single metric component. This choice may be obtained directly from Eq. (2.1) by a suitable conformal transformation combined with a redefinition of the radial coordinate. In this gauge $R^0_0 = R^z_z = 0$ [1] so it cannot contain the well-known solutions with a cosmological constant of Eq. (2.2). The cylindrical version of the "Mannheim gauge"[1] $M(r) = 1/B(r)$, $L(r) = r$ with $K(r) = B(r)$, turns out to be complicated and the simplifications from spherical symmetry are not observed.

An alternative parametrization of the metric tensor which solves these difficulties is the following gauge choice

$$M = 1 \quad , \quad L = \frac{dB}{dr} = B' \quad , \quad K(r) = B(r) \tag{2.4}$$

where we also resort to dimensionless coordinates. This metric is equivalent to that of the "Mannheim gauge", but has the advantage that it leads to autonomous equations which we can solve by quadrature.

Note however, that this gauge excludes the "flat" $\Lambda = 0$ solutions. Conformally flat solutions are of course still allowed.

In this gauge the components of Ricci tensor and scalar take the form

$$R^0_0 = R^z_z = -\left(\frac{B'}{B}\right)^2 - 2\frac{B''}{B} \ , \ R^r_r = R^\varphi_\varphi = -\frac{B'''}{B'} - 2\frac{B''}{B} \ , \ R = -2\left[\left(\frac{B'}{B}\right)^2 + 4\frac{B''}{B} + \frac{B'''}{B'}\right] \tag{2.5}$$

while the rest vanish.

Already at this stage we can obtain easily the constant Ricci solutions of Eq. (2.2). It is enough to solve the (00) equation which is readily integrated to give

$$\frac{1}{2}(B')^2 + \frac{\Lambda}{24}B^2 - \frac{b}{B} = 0 \tag{2.6}$$

where b is the integration constant. This is a trivial "mechanical" equation whose solutions are easily obtained by inspection. We will discuss these solutions within the more general framework in the next section.

The non-zero components of the Weyl tensor are all proportional to a single quantity namely

$$C^{0r}{}_{0r} = C^{0\varphi}{}_{0\varphi} = C^{rz}{}_{rz} = C^{\varphi z}{}_{\varphi z} = \mathcal{C}/6 \quad , \quad C^{r\varphi}{}_{r\varphi} = C^{0z}{}_{0z} = -\mathcal{C}/3 \tag{2.7}$$

where

$$\mathcal{C} = \frac{B'''}{B'} - 2\frac{B''}{B} + \left(\frac{B'}{B}\right)^2 , \tag{2.8}$$

[1] see the discussion about the metric $\hat{g}_{\mu\nu} = diag(1, -1, -H^2, -1)$ at the end of this section

and its square is given simply by $C_{\kappa\lambda\mu\nu}C^{\kappa\lambda\mu\nu} = 4\mathcal{C}^2/3$.

the non-vanishing components of the Bach tensor are

$$W^0_0 = W^z_z = \frac{B^{(5)}}{3B'} + \frac{2B^{(4)}}{3B} - \frac{B^{(4)}B''}{3B'^2} + \frac{B'''B''^2}{3B'^3} + \frac{2B'''B''}{3BB'} - \frac{2B'''B'}{3B^2} - \frac{2B'''^2}{3B'^2} - \frac{4B''^2}{3B^2} + \frac{4B'^2B''}{3B^3} - \frac{B'^4}{3B^4} \tag{2.9}$$

$$W^r_r = \frac{2B^{(4)}}{3B} - \frac{2B^{(4)}B''}{3B'^2} + \frac{2B'''B''^2}{3B'^3} - \frac{2B'''B''}{BB'} + \frac{2B'''B'}{3B^2} + \frac{B'''^2}{3B'^2} + \frac{4B''^2}{3B^2} - \frac{4B'^2B''}{3B^3} + \frac{B'^4}{3B^4} \tag{2.10}$$

while the fourth one, W^φ_φ can be obtained immediately from the identity $W^\mu_\mu = 0$.

Using an exponential transformation $B = e^\beta$ simplifies the expressions a little. W^r_r and the "Weyl tensor quantity" $\mathcal{C}$ which will be used in the following, become:

$$W^r_r = -\beta''^2 + \frac{\beta^{(3)2}}{3\beta'^2} + \frac{2\beta^{(3)}\beta''^2}{3\beta'^3} - \frac{4\beta^{(3)}\beta''}{3\beta'} - \frac{2\beta^{(4)}\beta''}{3\beta'^2} \quad , \qquad \mathcal{C} = \frac{\beta'''}{\beta'} + \beta'' \tag{2.11}$$

An easy way to obtain W^ν_μ is to start with the line element $ds^2 = B^2[dt^2 - d\varrho^2 - H^2 d\varphi^2 - dz^2]$ whose Bach tensor is given by $W^\nu_\mu = \hat{W}^\nu_\mu/B^4$ where $\hat{W}^\nu_\mu$ is calculated from the metric $\hat{g}_{\mu\nu} = diag(1, -1, -H^2, -1)$. The components $\hat{W}^\nu_\mu$ are relatively easy to obtain since $\hat{g}_{\mu\nu}$ has only one non-vanishing independent Riemann component: $\hat{R}_{\varrho\varphi\varrho\varphi} = -H d^2H/d\varrho^2$. We thus get

$$\hat{W}^0_0 = \hat{W}^z_z = \frac{1}{3}\left[\frac{H''''}{H} - \frac{H'H'''}{H^2} - 2\left(\frac{H''}{H}\right)^2 + \frac{H'^2H''}{H^3}\right] ,$$

$$\hat{W}^\varrho_\varrho = \frac{1}{3}\left[-2\frac{H'H'''}{H^2} + \left(\frac{H''}{H}\right)^2 + 2\frac{H'^2H''}{H^3}\right], \qquad ' = \frac{d}{d\varrho} \tag{2.12}$$

while $\hat{W}^\varphi_\varphi$ can be obtained from $W^\mu_\mu = 0$. A further coordinate transformation $Bd\varrho = dr$ together with $H = d\beta/dr$ yields after lengthy calculations the above components of the Bach tensor of Eqs. (2.9)-(2.10). As a check we have verified that these components satisfy a covariant conservation law, $\nabla_\mu W^\mu_\nu = 0$ which in the present case reduces to the single equation

$$BB'(W^r_r)' + 2(BB'' + B'^2)W^r_r + 2(BB'' - B'^2)W^0_0 = 0. \tag{2.13}$$

We also calculated W^φ_φ explicitly and verified that indeed $W^\mu_\mu = 0$.

3 String-Like Solutions

Solving the equations becomes possible thanks to the following identity

$$W^r_r = -\frac{2}{3}\frac{e^{-3\beta/2}}{\beta'}\left[\beta''\left(e^{3\beta/2}\left(\beta'' + \frac{\beta'''}{\beta'}\right)\right)' - \frac{1}{2}\beta'''e^{3\beta/2}\left(\beta'' + \frac{\beta'''}{\beta'}\right)\right] \tag{3.1}$$

which may be written as a total derivative in the following form:

$$W^r_r = -\frac{2\epsilon}{3}\frac{(\epsilon\beta'')^{3/2}\,e^{-3\beta/2}}{\beta'}\left[\frac{e^{3\beta/2}}{\sqrt{\epsilon\beta''}}\left(\beta''+\frac{\beta'''}{\beta'}\right)\right]' = \frac{4}{3}\frac{(\epsilon\beta'')^{3/2}\,e^{-3\beta/2}}{\beta'}\left[\frac{\left(\sqrt{\epsilon e^{\beta}\beta''}\right)'}{(e^{-\beta})'}\right]' \tag{3.2}$$

where $\epsilon = \pm 1$ according to whether β'' is positive or negative. The motivation to look for such identities comes from the fact that the components of Bach tensor may be expressed in terms of the Weyl tensor as in Eq. (1.4). Although it may seem simpler to get solutions in other gauges like the Mannheim gauge or in terms of the metric component $H(r)$ defined above, we were able to integrate the equations in terms of $B(r)$ (or $\beta(r)$) only.

Solving now the equation $W^r_r = 0$ is straightforward and quite simple. We will use both β and B alternatively according to convenience. The first kind of solutions satisfies $\beta'' = 0$ which is solved by

$$B(r) = B_0 e^{kr} \tag{3.3}$$

where k may be either positive or negative.

The only other possibility is that the ratio $(\sqrt{\epsilon e^{\beta}\beta''})'/(e^{-\beta})'$ in the second factor of (3.2) is a constant, say c, so we obtain after two integrations the following "mechanical" equation

$$\frac{1}{2}(\beta')^2 + \frac{\epsilon}{3c}(a + ce^{-\beta})^3 = E \;\;,\;\; c \neq 0 \quad ; \qquad \frac{1}{2}(\beta')^2 + \epsilon a^2 e^{-\beta} = E \;\;,\;\; c = 0 \tag{3.4}$$

where a and E are integration constants. Note that the special case $c = 0$ needs special care. A special family of solutions of this case is the above (3.3). Others will be considered later. Actually, it is easy to see from Eqs (2.11) and (3.2) that $c = 0$ corresponds to conformally flat solutions with vanishing Weyl tensor.

We therefore define an "effective potential" $V_{eff}(\beta)$ which satisfies $(\beta')^2/2 + V_{eff}(\beta) = E$:

$$V_{eff}(\beta) = \begin{cases} \frac{\epsilon}{3c}(a + ce^{-\beta})^3 & ,\;\; c \neq 0 \\ \epsilon a^2 e^{-\beta} & ,\;\; c = 0\,. \end{cases} \tag{3.5}$$

Fig. 1 presents the general behavior of $V_{eff}(\beta)$ for $\epsilon = +1$. The curves for $\epsilon = -1$ are just the same taken "upside-down". In terms of the metric function B the equation becomes $(B')^2/2 + V_{eff}(B) = 0$ where now

$$V_{eff}(B) = \begin{cases} (\frac{\epsilon a^3}{3c} - E)B^2 + \epsilon a^2 B + \frac{\epsilon c^2}{3B} + \epsilon ac & ,\;\; c \neq 0 \\ \epsilon a^2 B - EB^2 & ,\;\; c = 0\,. \end{cases} \tag{3.6}$$

The solutions may be expressed directly in terms of the metric function B, but part of the presentation is clearer in terms of β. So we will use both forms of solutions

$$r(B) = \int dB/\sqrt{-2V_{eff}(B)} \quad ; \quad r(\beta) = \int d\beta/\sqrt{2\,(E - V_{eff}(\beta))} \tag{3.7}$$

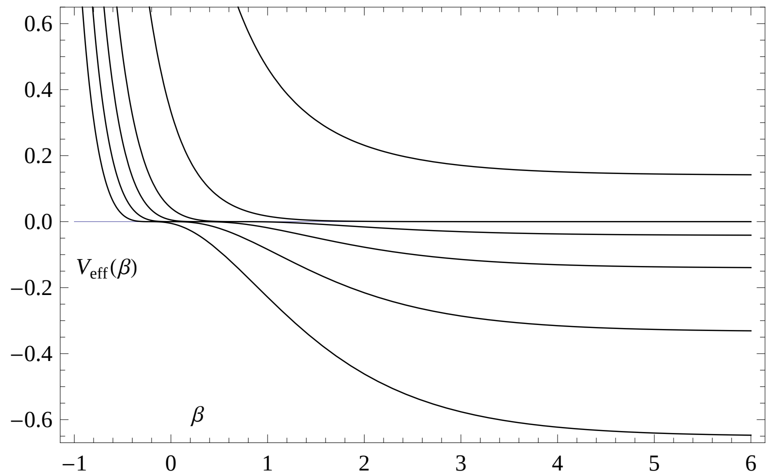


Figure 1: The effective potential $V_{eff}(\beta)$ for $\epsilon = 1$, $c = 1$ and $a = -1.25, -1, -0.75, -0.5, 0, 0.75$. In order to identify the curves, note that the asymptotic value of $V_{eff}(\beta)$ increases with a/c for $\epsilon = 1$.

where the integration limits are determined by the boundary conditions.

By inspection of the potential function one can conclude that there are 3 distinct types of solutions: one for $\epsilon = +1$ and two for $\epsilon = -1$. It is easy to see that for $\epsilon = +1$ there are solutions only for values of E obeying $E > a^3/3c$ (or $E > 0$ for $c = 0$). The effective potential $V_{eff}(\beta)$ decreases monotonically with β and the solutions have a finite minimal value unless $c = 0$. On the other hand, for $\epsilon = -1$, E can have any real value. This case splits therefore according to whether E is above or below the maximal value of the effective potential $V_{eff}(\beta)$ which is $-a^3/3c$ (or 0). If $E < -a^3/3c$, β or B is bounded from above. If $E \geq -a^3/3c$, the solutions can have the whole range $0 \leq B < \infty$. For future use we define the "energy excess" parameter ζ by $3E/c^2 = \zeta^3 + \epsilon(a/c)^3$.

It is possible to express the solutions in terms of hyperelliptic integrals, but since the effective potential is monotonic, the possible behaviors of the solutions are limited so it is enough to introduce the following two kinds of real functions:

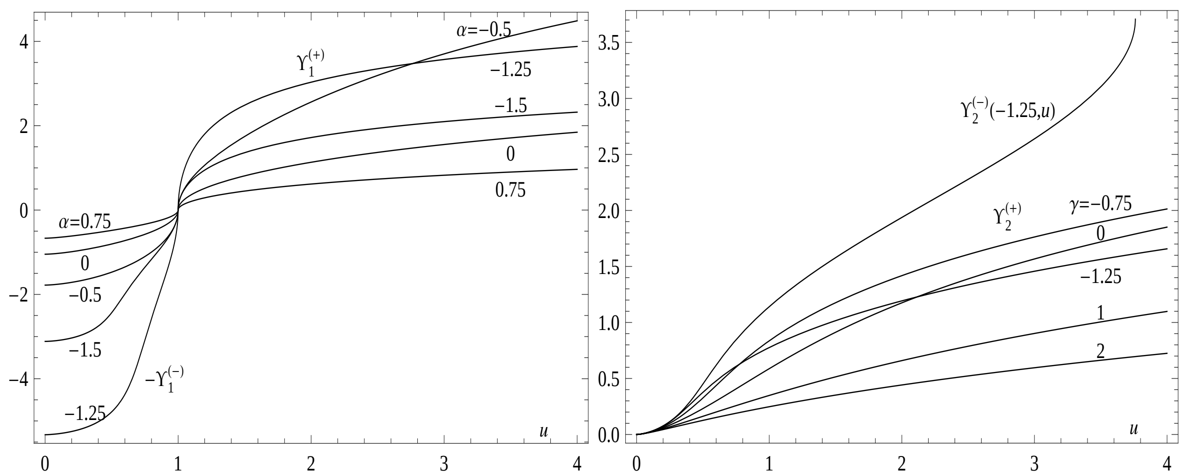
$$\Upsilon_1^{(+)}(\alpha, u) = \int_1^u \frac{dt}{t\sqrt{(\alpha+1)^3 - (\alpha+1/t)^3}} \quad , \quad u \geq 1 \tag{3.8}$$

$$\Upsilon_1^{(-)}(\alpha, u) = \int_u^1 \frac{dt}{t\sqrt{(\alpha+1/t)^3 - (\alpha+1)^3}} \quad , \quad 0 \leq u \leq 1 \tag{3.9}$$

and

$$\Upsilon_2^{(+)}(\gamma, u) = \int_0^u \frac{dt}{t\sqrt{(\gamma+1/t)^3 - \gamma^3 + 1}} \quad , \quad u \geq 0 \tag{3.10}$$

$$\Upsilon_2^{(-)}(\gamma, u) = \int_0^u \frac{dt}{t\sqrt{(\gamma+1/t)^3 - \gamma^3 - 1}} \quad , \quad 0 \leq u \leq u_{max} \tag{3.11}$$



(a) (b)

Figure 2: Plots of the Υ-functions: (a) $\Upsilon_1^{(\pm)}(\alpha, u)$; the $(+,-)$ correspond to $u \geq 1$, $u \leq 1$ respectively. (b) $\Upsilon_2^{(\pm)}(\gamma, u)$.

where u_{max} is the solution of $(\gamma + 1/u)^3 = \gamma^3 + 1$, that is:

$$u_{max} = \frac{1}{\sqrt[3]{1+\gamma^3} - \gamma} \tag{3.12}$$

with the cube root $\sqrt[3]{x}$ means $-|x|^{1/3}$ for $x < 0$.

Figure 2 shows the typical behavior of the Υ-functions. Note that there are only three independent Υ-functions: $\Upsilon_1^{(-)}(\alpha, u)$ and $\Upsilon_2^{(-)}(\gamma, u)$ are actually the same up to simple rescaling and translation:

$$\Upsilon_1^{(-)}(\alpha, 0) - \Upsilon_1^{(-)}(\alpha, u) = \frac{1}{\zeta_0^{3/2}} \Upsilon_2^{(-)}(\frac{\alpha}{\zeta_0}, \zeta_0 u) \ ; \ \zeta_0(\alpha) = (1 + 3\alpha + 3\alpha^2)^{1/3} \tag{3.13}$$

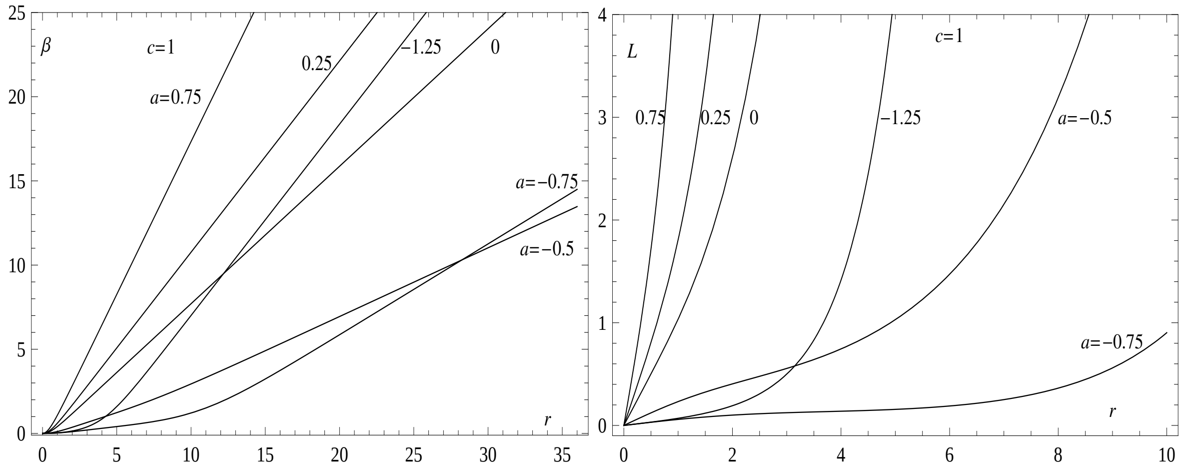
In the special cases $\alpha = 0$ and $\gamma = 0$ (which correspond to $a = 0$) the Υ-functions get the elementary forms:

$$\begin{aligned} \Upsilon_1^{(+)}(0, u) &= \frac{2}{3}\cosh^{-1}(u^{3/2}) \ , \quad \Upsilon_1^{(-)}(0, u) = \frac{2}{3}\cos^{-1}(u^{3/2}) \\ \Upsilon_2^{(+)}(0, u) &= \frac{2}{3}\sinh^{-1}(u^{3/2}) \ , \quad \Upsilon_2^{(-)}(0, u) = \frac{2}{3}\sin^{-1}(u^{3/2}). \end{aligned} \tag{3.14}$$

These correspond to the constant Ricci solutions that were mentioned in the previous section following Eq. (2.6). The fact that $a = 0$ corresponds to constant Ricci solutions can be also inferred by noticing that in this case the potential function $V_{eff}(B)$ of Eq. (3.6) reduces to that in Eq. (2.6).

Two of the three kinds of the solutions of Eq. (3.4) can be expressed in terms of the Υ_1 functions as

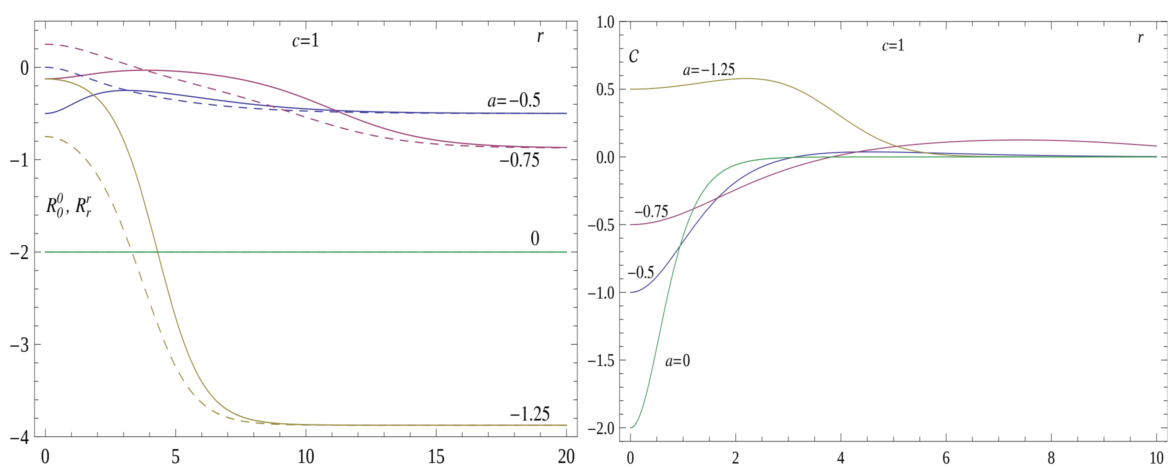
$$\sqrt{\frac{2}{3}} \frac{|c|}{B_0^3}(r - r_0) = \Upsilon_1^{(\pm)}(\frac{a}{c}B_0, \frac{B}{B_0}) \tag{3.15}$$



(a) (b)

Figure 3: Typical solutions for $\epsilon = 1$. Remember that $L = B'$. The other parameters are $c = 1$ and $a = -1.25, -0.75, -0.5, 0, 0.25, 0.75$. Note the non-monotonic dependence of the β-slope on a.

where the $\pm$ corresponds to the ϵ value. The $\epsilon = +1$ solutions are open with a minimum at $B = B_0$ while the $\epsilon = -1$ ones are closed (that is bounded from above) for $E < -a^3/3c$. Note that in this case $B(r)$ can be extended by "reflection" with respect to the $r = r_0$ line, but $L(r) = B'$ which vanishes there, reduces the domain of the solutions such that $B(r)$ is monotonic. We will not elaborate on this kind of solutions further.



(a) (b)

Figure 4: (a) The Ricci components R^0_0 (solid) and R^r_r (dashed). (b) The Weyl tensor component $\mathcal{C}$. All plots for $\epsilon = 1$, $c = 1$ and $a = -1.25, -0.75, -0.5, 0$. These are 4 of the solutions shown in Fig. 3. The curves for the other two cases are similar, but have much more negative Ricci components.

Fig. 3 shows typical solutions for $\epsilon = 1$ with the boundary condition $B(0) = 1$ and $B'(0) = 0$. The components of the corresponding Ricci tensor are presented in Fig. 4. They are all compatible with the asymptotic condition (2.3) as should be the case. Indeed, it is easy to find from (2.5) that

$$\kappa = -8c^2\zeta^3 \tag{3.16}$$

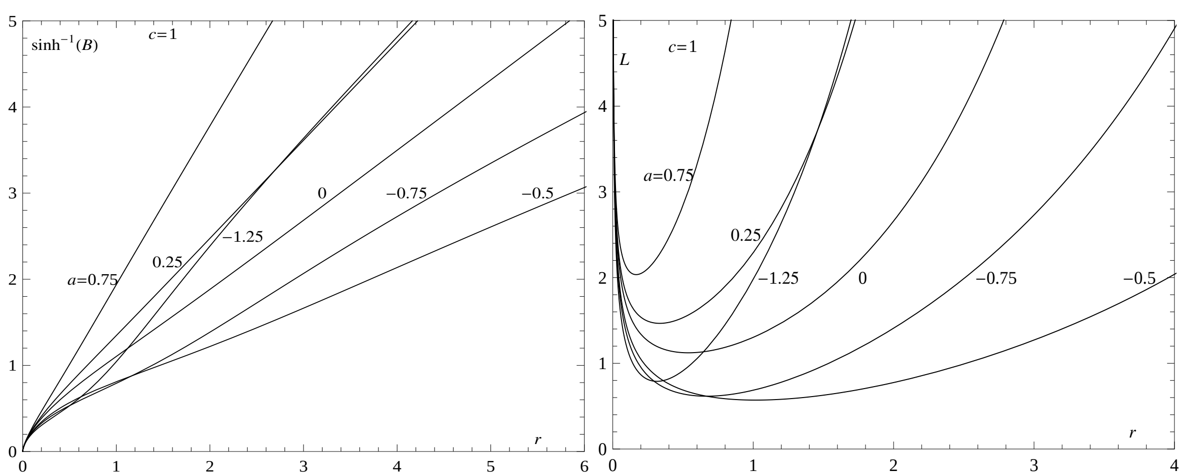
where ζ is the "energy excess" parameter defined above. Since $\zeta^3 > 0$ for the open solutions, all of them are asymptotically "negatively curved" (that is $\kappa < 0$).

A special member of this family is the AdS soliton which corresponds to $a = 0$ and has constant Ricci components, as it solves Eq. (2.2) or (2.6). Note however, that unlike (A)dS space, it is not conformally flat, as is also obvious from the corresponding curve in Fig. 4b.

The third type of solutions is with $\epsilon = -1$ and $E \geq -a^3/3c$, so they can have the whole range $0 \leq B < \infty$. It may be written simply using the energy excess parameter ζ which is positive in this case too:

$$\sqrt{\frac{2}{3}}|c|\zeta^{3/2}(r - r_0) = \Upsilon_2^{(+)}(\frac{a}{c\zeta}, \zeta B) \tag{3.17}$$

Fig. 5 shows typical solutions of this kind where we choose $B(0) = 0$. Fig. 6 has the components of the corresponding Ricci tensor which shows a singularity on the axis. However, these solutions may still be physically relevant as exterior solutions of appropriate cylindrically-symmetric sources. We took the same values as in Fig. 3 for the parameters a and c as well as for the "energy excess" ζ. We also chose to present $\sinh^{-1}(B)$ instead of B in order to cover both the small B and large B regions. The exponential increase of $B(r)$ away from the



(a) (b)

Figure 5: Typical open solutions for $\epsilon = -1$. Remember that $L = B'$. The other parameters are $c = 1$ and $a = -1.25$, -0.75, -0.5, 0, 0.25, 0.75. Note again the non-monotonic dependence of the slopes on a.

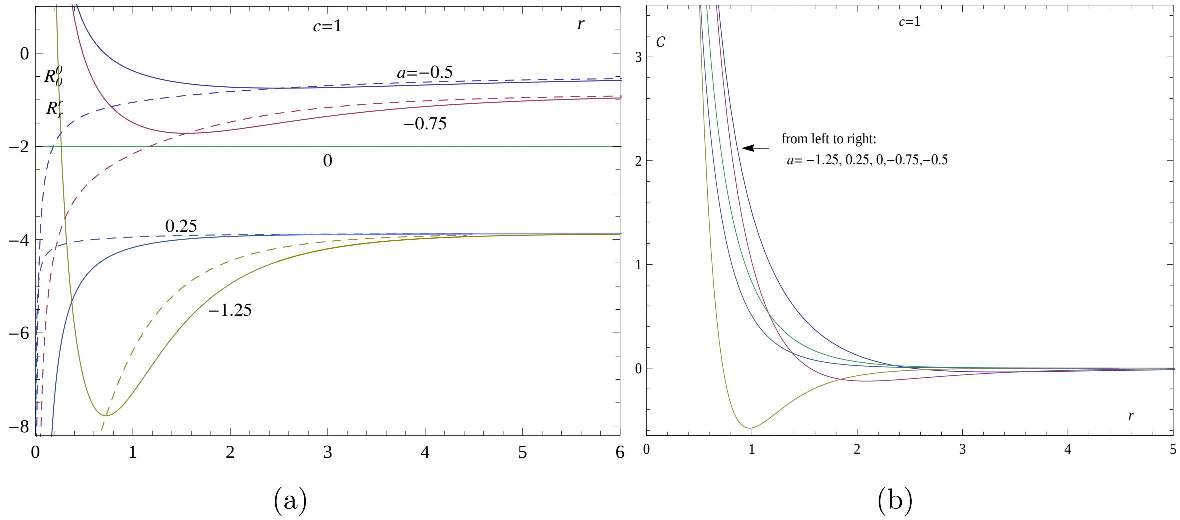


Figure 6: (a) The Ricci components R_0^0 (solid) and R_r^r (dashed). (b) The Weyl tensor component $\mathcal{C}$. All plots for $\epsilon = -1$, $c = 1$ and $a = -1.25$, -0.75, -0.5, 0, 0.25. These are 5 of the solutions shown in Fig. 5. The $a = 0.75$ curves are similar to the $a = 0.25$ ones, but have much more negative Ricci components.

axis is obvious. An exception from the generic exponential behavior is the minimal E-value for unbounded solutions which correspond to $\zeta = 0$. In this case the asymptotic behavior is $B \sim r^2$. The asymptotic behavior of the Ricci components is given again by (2.3) with (3.16) so in both cases $\kappa < 0$. The $\zeta = 0$ solution is asymptotically Ricci flat.

Note that again the special case $a = 0$ allows for explicit (constant Ricci) solutions in terms of elementary functions - see (3.14). For the "last unbounded trajectory", which corresponds to $\zeta = 0$ (with $\epsilon = -1$) we find $B = [\sqrt{3/2}|c|(r - r_0)]^{2/3}$. This solution is not only asymptotically Ricci flat, but has $R_{\mu\nu} = 0$ identically and is actually the well-known "Kasner" string-like solution of ordinary GR. Another special case which requires anyhow a separate treatment is $c = 0$. In this case we can just integrate Eq. (3.7) directly to get the general solutions

$$B = B_0 \cosh^2\left(\frac{|a|(r - r_0)}{\sqrt{2B_0}}\right) \ , \quad \epsilon = 1 \tag{3.18}$$

$$B = B_0 \cos^2\left(\frac{|a|(r - r_0)}{\sqrt{2B_0}}\right) \ , \quad \epsilon = -1 \ , \quad E < 0$$

instead of (3.15), while instead of (3.17) we have

$$B = \frac{a^2}{E} \sinh^2\left(\sqrt{E/2} \ \ (r - r_0)\right) \ , \quad \epsilon = -1 \ , \quad E > 0 \tag{3.19}$$

The special case $E = 0$ can be obtained from the limit of this equation to give a behavior of

$$B = a^2 (r - r_0)^2 / 2 \ , \quad \epsilon = -1 \ , \quad E = 0 \tag{3.20}$$

This family is supplemented by the exponential solution (3.3) encountered already at the beginning.

All these $c = 0$ solutions have non-trivial curvatures, but as mentioned above they are conformally flat. However, they are not trivial even within the framework of CG, since they are either conformal to Minkowski space-time with a conic angular deficit like (3.20), or to a certain region of it like the others. The solution (3.20) is therefore the analog in this gauge of CG to the conical string-like solution of GR.

Finally we note a possible generalization of this work to the scalar-tensor extension of CG which allows for a breakdown of the conformal symmetry. Some cylindrically-symmetric solutions of this kind were obtained numerically in a recent study [12], but considering the results presented in this section, it may turn out possible to obtain explicit exact solutions in the gauge where the scalar field is constant. The metric tensor will have now two independent components, but the second may be taken as a conformal factor (as in ref [7]) which will simplify the equations so that analytical treatment will be possible.

4 Null Geodesics

We proceed here to examine further the nature of these solutions by studying their geodesics. The geodesic equations in the metric $diag(B^2, -1, -L^2, -B^2)$ are easily integrated to give

$$\dot{z} = k \ , \quad L^2\dot{\varphi}/B^2 = \ell \ , \quad \dot{r}^2 + (k^2-1)B^2 + \ell^2 B^4/L^2 = -\mu B^4/\mathcal{E}^2 \tag{4.1}$$

where the coordinates are functions of time t and $k < 1$, ℓ and $\mathcal{E}$ are constants of the motion. The parameter $\mu = 0, 1$ distinguishes between lightlike or timelike geodesics respectively. Since only null geodesics have an invariant meaning, we will concentrate on them.

One can analyze numerically the trajectories by the effective potential for the r-motion which satisfies $\dot{r}^2/2 + U_{eff}(r) = 0$. However, analytic treatment is possible if we "compromise" on a parametric representation of the solutions $r(t)$ by $(r(B), t(B))$. We will not expect any trouble with that since we concentrate in the open solutions where $B(r)$ is monotonic. The r-equation will be replaced by $\dot{B}^2/2 + W_{eff}(B) = 0$ with

$$W_{eff}(B) = \left[\frac{\ell^2}{2} + (1-k^2)\left(\frac{\epsilon c^2\alpha^3}{3} - E\right)\right] B^4 + \epsilon c^2(1-k^2)\left(\alpha^2 B^2 + \alpha B + \frac{1}{3}\right) B \tag{4.2}$$

where $\alpha = a/c$. This is the potential for the general case with $c \neq 0$. The $c = 0$ case needs as usual a special treatment and is given by (4.8) below. Actually, since there exist explicit solutions in this case one may obtain also $U_{eff}(r)$ explicitly. It is straightforward to analyze the different trajectory types from $W_{eff}(B)$ and especially its zeroes. The solutions may be

bounded (in B or in r) where $W_{eff}(B) < 0$ in a finite interval or unbounded (or open) if this interval extends to infinity. The process is simplified by noticing that $W_{eff}(B)$ has only one zero in addition to the possible one at $B = 0$.

For $\epsilon = 1$ all solutions are open, namely $B_{min} \leq B < \infty$ where B_{min} depends on the constants of the motion k and ℓ and on the other geometrical parameters a, c and E (see below). A necessary condition for solutions to exist (for $\epsilon = 1$) is that the coefficient of the B^4 term in $W_{eff}(B)$ which is proportional (with the positive coefficient $c^2(1-k^2)$) to

$$\xi = \frac{3\ell^2}{2c^2(1-k^2)} - \frac{3E}{c^2} + \epsilon\alpha^3 \tag{4.3}$$

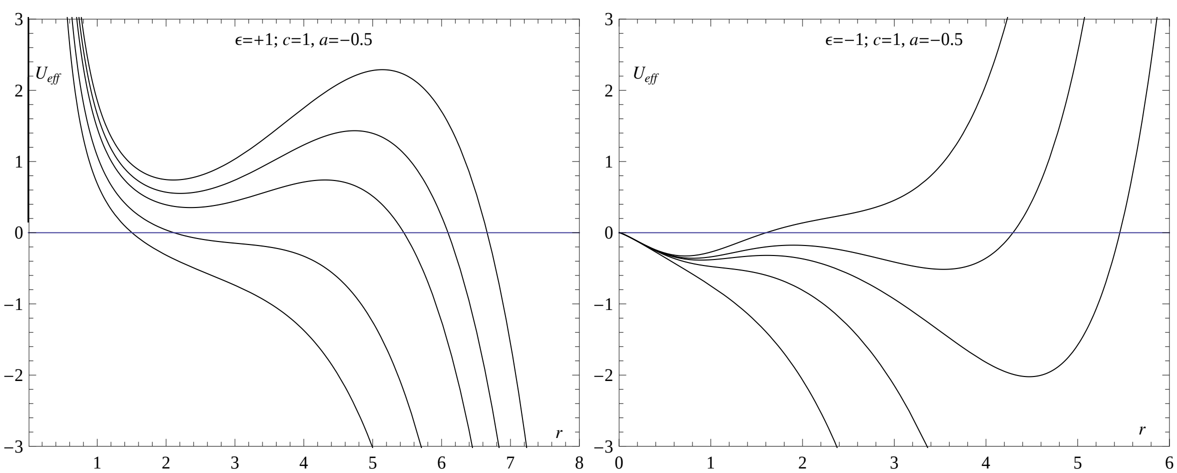
will be negative, otherwise there will be no zeroes of $W_{eff}(B)$ for $B \geq 1$ which is the domain of B in this case. This imposes an upper bound on the combination $\ell^2/(1-k^2)$:

$$\frac{\ell^2}{1-k^2} < 2\left(E - \frac{c^2\alpha^3}{3}\right) \tag{4.4}$$

For $\epsilon = -1$ there are two cases: open solutions which now extend for $0 \leq B < \infty$ occur if Eq. (4.4) is satisfied. In the complementary case we have bounded solutions for $0 \leq B < B_{max}$ where B_{max} depends on the characteristic parameters. B_{max} and also B_{min} can be written together as

$$B_{min/max} = \frac{\alpha^2}{\epsilon\xi}\left(-1 + \sqrt[3]{\epsilon\xi/\alpha^3 - 1} - \sqrt[3]{(\epsilon\xi/\alpha^3 - 1)^2}\right) \tag{4.5}$$

with the corresponding value of ϵ.



(a) (b)

Figure 7: Effective potential (divided by $(1-k^2)/2$) curves for null geodesics in the two types of metrics with $c = 1$ and $a = -0.5$. (a) $\epsilon = 1$, $\xi = -0.15, -0.13, -0.11, -0.10, -0.09$; (b) $\epsilon = -1$, $\xi = -0.2, 0, 0.07, 0.1, 0.15$. In both cases the higher curves correspond to the larger values of ξ.

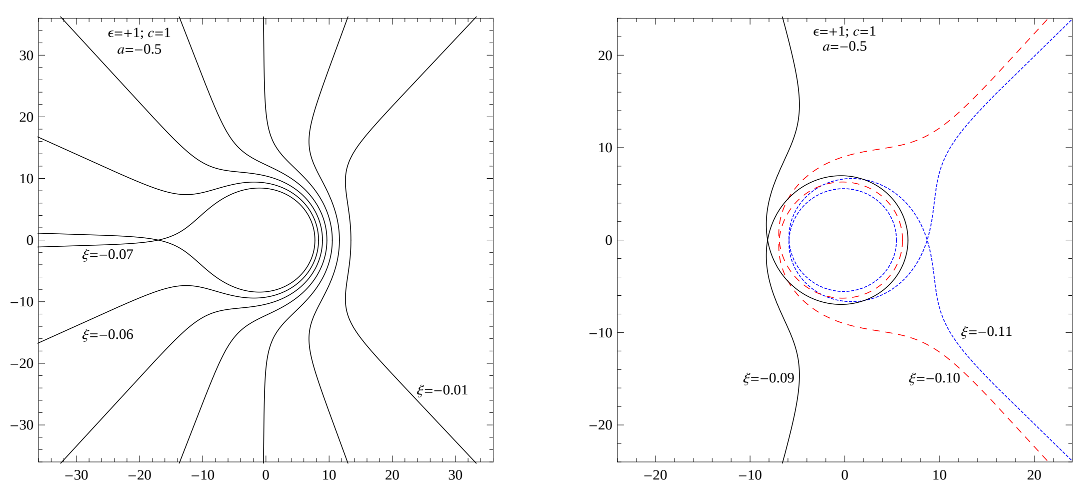
Plots of $U_{eff}(r)$ which present all these properties are shown in Fig 7. For $\epsilon = 1$ only $\xi < 0$ curves are shown since otherwise $U_{eff}(r)$ is always positive and no solutions exist. Note that each curve in this figure corresponds to a single null geodesic (up to t-translation) defined by k and ℓ since only vanishing "effective energy" is allowed.

In order to find out about the shape of the geodesics, one may solve for $r(\varphi)$ or equivalently for the parametric representation $(r(B), \varphi(B))$. It turns out to be simpler to transform to $v = 1/B$ where we find the following first order equation for light-like trajectories, or more accurately, their projection on the (r, φ) plane:

$$\frac{1}{2}\left(\frac{dv}{d\varphi}\right)^2 + 2\left[1 - \frac{2(1-k^2)}{\ell^2}\left(E - \frac{\epsilon c^2}{3}(\alpha + v)^3\right)\right]\left(E - \frac{\epsilon c^2}{3}(\alpha+v)^3\right)^2 = 0 \tag{4.6}$$

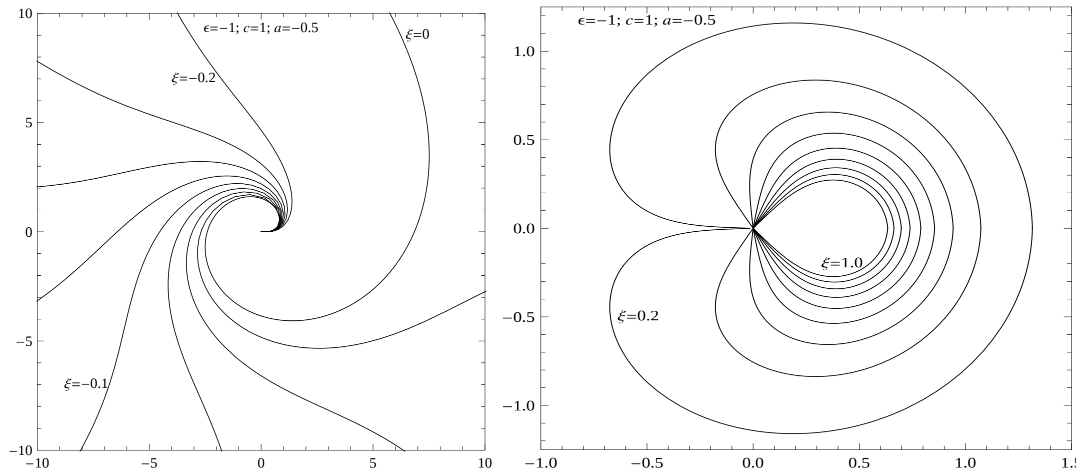
The projections of some typical trajectories appear in Figs. 8-9. They demonstrate explicitly the features obtained from the general discussion above. The $\epsilon = 1$ solutions are all open with a "periastron" chosen to be always at $\varphi = 0$. Several windings are possible according to the values of the parameters. The $\epsilon = -1$ solutions may be either open with $0 \le r < \infty$, or bounded with $0 \le r \le r_{max}$. Again, several windings are possible.

Finally we discuss shortly some special cases. The first is $a = 0$. Generally null geodesics in this case are just slightly different from the adjacent solutions except when also $\xi = 0$ for $\epsilon = -1$. This is the only case when $U_{eff}(r)$ vanishes asymptotically to allow open geodesics with $\dot{r}(t) \to 0$ as $t \to \infty$. The resulting orbits are spirals which become denser as r increases. Actually, the effective potential $U_{eff}(r)$ can be written explicitly in terms of elementary functions, and the



(a) (b)

Figure 8: Projection in the (r, φ) plane of null trajectories with $\epsilon = +1$, $c = 1$, $a = -0.5$. (a) Solutions with ξ-values from $\xi = -0.07$ to $\xi = -0.01$ in even steps; (b) Three more winding solutions with $\xi = -0.11,\ -0.10,\ -0.09$.



(a) (b)

Figure 9: Projection in the (r, φ) plane of null trajectories with $\epsilon = -1$, $c = 1$, $a = -0.5$. (a) Open solutions with ξ-values from $\xi = -0.2$ to $\xi = 0$ in even steps; (b) Bounded solutions with ξ-values from $\xi = 0.2$ to $\xi = 1$ in even steps.

typical null geodesics with $r(0) = 0$ and $\varphi(0) = 0$ are

$$r(t) = \sqrt{\frac{2}{3}}\frac{1}{c}\sinh^{-1}(\bar{t}^3) \ , \quad \varphi(t) = \frac{3\ell}{2c^3}\sqrt{\frac{6}{1-k^2}}\left[\bar{t} - \frac{1}{6}\left(\tan^{-1}(2\bar{t}-\sqrt{3}) + \tan^{-1}(2\bar{t}+\sqrt{3})\right)\right.$$
$$\left. -\frac{1}{3}\tan^{-1}\bar{t} + \sqrt{3}\log\left(\frac{\bar{t}^2 - \sqrt{3}\bar{t}+1}{\bar{t}^2+\sqrt{3}\bar{t}+1}\right)\right] \qquad (4.7)$$

where $\bar{t} = \sqrt{(1-k^2)/6}\; ct$.

The case $c = 0$ also allows to obtain explicit expressions for $U_{eff}(r)$, but it is easier to solve for $B(t)$ using the effective potential of Eq. (4.2) which now simplifies to

$$W_{eff}^{(c=0)}(B) = \left(\frac{\ell^2}{2} - (1-k^2)E\right)B^4 + \epsilon(1-k^2)a^2B^3 \qquad (4.8)$$

The transformation $v = 1/B$ gives a mechanical equation with a linear potential which gives easily the following solutions in the various cases. Two of them can be written as

$$\frac{1}{B} = -\frac{\epsilon(1-k^2)a^2t^2}{2} + \frac{\epsilon}{a^2}\left(E - \frac{\ell^2}{2(1-k^2)}\right) \qquad (4.9)$$

where we chose $t = 0$ to be the time which corresponds to the minimal or maximal radial distances for $\epsilon = 1$ or $\epsilon = -1$ respectively. For $\epsilon = 1$, t should be further restricted such that $1/B$ is non-negative. Note that the relative sizes of the two quantities in the bracket of the second term are such that it is always positive. The third kind of geodesics which have $0 < B < \infty$ exist only for $\epsilon = -1$ and are given by

$$\frac{1}{B} = \frac{(1-k^2)a^2t^2}{2} + \sqrt{2(1-k^2)E - \ell^2}\; t \qquad (4.10)$$

In order to get the explicit $r(t)$ dependence of these geodesics one needs the corresponding $B(r)$ solutions for this case - Eqs. (3.19)–(3.19). We will not present here this last step. The explicit solution $\varphi(t)$ which we skip too may be obtained by direct integration of (see (4.1))

$$\frac{d\varphi}{dt} = \frac{\ell B^2}{L^2} = \frac{\ell}{2(E - \epsilon a^2/B)} \tag{4.11}$$

with $1/B(t)$ obtained from (4.9)–(4.10) above.

5 Conclusion

We have analyzed the vacuum field equations of CG in the static cylindrically-symmetric case and found all solutions explicitly. In some cases the solutions are expressed in terms of elementary functions.

There are three kinds of solutions: two open ones and one closed. The open ones split into a regular family (with a possible conic singularity on the axis) which contains the AdS soliton and a family of singular solutions where g_{00} vanishes on the axis and the Ricci and Weyl components diverge there. These two families of open solutions have generically asymptotically vanishing Weyl tensor. Some special cases are conformal to conical space-times.

We have further analyzed the null geodesics in the two families of the open spacetimes. We found that the singular spacetimes support two kinds of null geodesics: open ones (that is $0 < r < \infty$) which are usually spirals when the angular momentum is below a certain maximal value, and bounded ones when the angular momentum is above the critical value. In the regular spacetimes only open geodesics exist for the same range of angular momentum. Unless the angular momentum vanishes, the orbits in this case avoid $r = 0$. Above the critical value no geodesics exist at all.

A possible future application of these results is to use these families of exact solutions to analyze light bending in the vicinity of localized linear sources in CG. The analogous problem of light bending in spherically-symmetric gravitational field was only recently settled [19].

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